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On a Theoretically Conformable Duality for Semicontinuity of Set-Valued Mappings

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Abstract. The paper presents a study of semicontinuity of set-valued maps. The Berge definitions of upper and lower semicontinuities of set-valued maps are improved into weaker and slightly strong conditions, respectively, which have a theoretically conformable duality of semicontinuity of set-valued maps. The improved definitions of upper and lower semicontinuities are defined in terms of both neighborhoods and sequences in a metric space. As the result of this research, we understand the reason why certain conditions are needed to guarantee the equivalence between Berge's upper semicontinuity and Hogan's upper semicontinuity.

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1. Introduction

A set-valued map (point-to-set map or multifunction) from a set X into a set Y is a map which associates a subset of Y with each point of X. The notion of semicontinuity for a set-valued map is very important in the area of optimization as well as in other fields of applied mathematics. In particular, upper semicontinuity of a set-valued map is indispensable for fixed-point theorems and stability theory in mathematical programming.

The concepts of semicontinuous maps have been introduced in 1932 by G. Bouligand and K. Kuratowski; see [1] and the references therein. There are various different definitions of upper semicontinuity and lower semicontinuity so far, and they are classified into two categories; those defined in terms of neighborhoods and those defined in terms of sequences; see [1, 2, 3, 4] and [6, 7]. They are not always equivalent with each other without any conditions. Nevertheless, upper semicontinuity is considered to be the dual concept of lower semicontinuity in general. This is an incomplete point for theoretical duality, because Berge's upper

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semicontinuity and Hogan's upper semicontinuity are not coincident and Berge's lower semicontinuity and Hogan's lower semicontinuity are coincident in a metric space [3, 4] although those upper semicontinuities are considered to be the dual concepts of those lower semicontinuities, respectively. Hogan's upper semicontinuity and lower semicontinuity are called closed and open, and defined in terms of sequences in his paper [4], respectively. This paper points out the reason which causes the discrepancy between Berge's upper semicontinuity and Hogan's upper semicontinuity.

The purpose of this paper is to extend classical concepts of upper semicontinuity of set-valued maps and then to give a theoretical conformable duality between upper semicontinuity and lower semicontinuity. Also, this paper gives, in terms of sequences, equivalent definitions for the improved definitions of upper semicontinuity and lower semicontinuity.

The organization of this paper is given as follows. In Section 2, we points out the difference between Berge's upper semicontinuity and Hogan's upper semicontinuity. This difference is based on both the existence of pathological neighborhoods including an image set of a set-valued map and the possibility of an unbounded graph of a set-valued map. If a set-valued map F is uniformly compact near x and if F is closed (Hogan's upper semicontinuous) at x, then F is (Berge's) upper semicontinuous at x. Conversely, if $F(x)$ is a closed set and if F is (Berge's) upper semicontinuous at x, then F is closed (Hogan's upper semicontinuous) at x. Those two upper semicontinuities are extended into weaker notions, and then relationship among all of them is given. Moreover, we give an equivalent definition for the weakest one in the extended upper semicontinuities in terms of sequences. In Section 3, we observe improved notions of Berge's lower semicontinuity and Hogan's lower semicontinuity corresponding to extended upper semicontinuities, and then we give a theoretical conformable duality between upper semicontinuity and lower semicontinuity. These improved semicontinuities are defined in terms of both neighborhoods and sequences in a metric space.

2. Extensions of Upper Semicontinuity of Set-Valued Maps

Let X and Y be two topological spaces, and $F : X \to 2^Y$. The definitions of Berge's semicontinuities are meaningful in any topological spaces X and Y, but Hogan's semicontinuities are meaningful in any spaces X and Y where the concept of convergence is defined in terms of nets or sequences. Since notions of semicontinuity having a theoretical conformable duality between upper semicontinuity and lower semicontinuity are defined in terms of open balls with radius $\varepsilon > 0$, we shall assume that Y is a metric space in those definitions.

First, we begin with classical Berge's semicontinuities of set-valued maps.

Definition 2.1. (Berge's u.s.c.) A set-valued map $F : X \to 2^Y$ is said to be upper semicontinuous (u.s.c. for short) at x_0 if for any open set U with $F(x_0) \subset U$, there exists a neighborhood V of x_0 such that $F(x) \subset U$ for all $x \in V$.

Definition 2.2. (Berge's l.s.c.) A set-valued map $F : X \to 2^Y$ is said to be lower semicontinuous (l.s.c. for short) at x_0 if for any open set U with $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood V of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in V$.

The notions of upper semicontinuity and lower semicontinuity are distinct and not equivalent in general except on residual sets in a complete separable metric space Y ([1, Th.1.4.13]). We provide the following example.

Example 2.1. Let $X = Y = R$ and F_1, F_2 set-valued maps from R into 2^R defined by

$$F_1(x) := \begin{cases} \{y \in R \mid 0 \leq y \leq 1\} & \text{for } x < 0; \\ \{y \in R \mid 0 \leq y \leq x + 2\} & \text{for } x \geq 0, \end{cases}$$

$$F_2(x) := \begin{cases} \{y \in R \mid 0 \leq y < 1\} & \text{for } x \leq 0; \\ \{y \in R \mid 0 \leq y < x + 2\} & \text{for } x > 0. \end{cases}$$

One can verify that F_1 is u.s.c. at $x = 0$ but not l.s.c. at the point, and that F_2 is l.s.c. at $x = 0$ but not u.s.c. at the point.

In 1973, Hogan [4] gave an alternative to the Berge's semicontinuities of set-valued maps in the setting of open and closed maps. He also presented the relationship between Berge's semicontinuities and his ones; see [3, 4].

Definition 2.3. (Hogan's u.s.c.) Let X and Y be two metric spaces. A set-valued map $F : X \to 2^Y$ is said to be closed at x_0 if for any sequences $\{x_n\}$ with $x_n \to x_0$ and $\{y_n\}$ with $y_n \in F(x_n)$, $y_n \to y_0$ for some $y_0 \in Y$ implies that $y_0 \in F(x_0)$.

Definition 2.4. (Hogan's l.s.c.) Let X and Y be two metric spaces. A set-valued map $F : X \to 2^Y$ is said to be open at x_0 if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $y_0 \in F(x_0)$, there exists a sequence $\{y_n\}$ such that $y_n \in F(x_n)$ and $y_n \to y_0$ (i.e., $d_Y(y_\lambda, y_0) \to 0$).

The map F_1 in Example 2.1. is closed at $x = 0$ but not open at the point, and the map F_2 in Example 2.1. is open at $x = 0$ but not closed at the point. As known from this, the notions of closedness and openness have meanings similar to upper and lower semicontinuities, respectively. Actually, openness and lower semicontinuity are coincident, but closedness and upper semicontinuity are not equivalent; see [4, Th.1], [1, p.39], and the following example.

Example 2.2. Let $X = Y = R$ and F_3, F_4 set-valued maps from R into 2^R defined by

$$F_3(x) := \{y \in R \mid 0 \leq y < 1\},$$

$$F_4(x) := \begin{cases} \{y \in R \mid 0 \leq y \leq 1\} & \text{for } x \leq 0; \\ \{y \in R \mid \frac{1}{x} \leq y \leq \frac{1}{x} + 1\} & \text{for } x > 0. \end{cases}$$

One can verify that F_3 is u.s.c. at $x = 0$ but not closed at the point, and that F_4 is closed at $x = 0$ but not u.s.c. at the point.

We shall observe the reason why this discrepancy is caused although upper semicontinuity and closedness are considered to be the dual concepts of lower semicontinuity and openness, respectively. To this end, we extend the two notions into weaker ones, and then present the relationship among all of them.

Definition 2.5. (w-u.s.c.) A set-valued map $F : X \to 2^Y$ is said to be weakly upper semicontinuous (w-u.s.c. for short) at x_0 if for any open set U with $\mathbf{cl}\ F(x_0) \subset U$, there exists a neighborhood V of x_0 such that $F(x) \subset U$ for all $x \in V$.

Of course, an upper semicontinuous map is also weakly upper semicontinuous. Conversely, if F is weakly upper semicontinuous at x_0 and $F(x_0)$ is a closed set, then it is upper semicontinuous at the point. Weakly upper semicontinuity is a slight extension of upper semicontinuity, which takes in some pathological non-u.s.c. maps.

Example 2.3. Let $X = Y = R$ and F_5 a set-valued map from R into 2^R defined by

$$F_5(x) := \begin{cases} \{y \in R \mid 0 \leq y < 1\} & \text{for } x \leq 0; \\ \{y \in R \mid 0 \leq y < x + 1\} & \text{for } x > 0. \end{cases}$$

One can verify that F_5 is weakly u.s.c. at $x = 0$ but not u.s.c. at the point.

However, there is another example of a pathological set-valued map which is similar to an u.s.c. map in image values but not u.s.c.

Example 2.4. Let $X = R_+$, $Y = R^2$, and F_6 a set-valued map from R_+ into 2^{R^2} defined by

$$F_6(x) := \left\{ (z_1, z_2) \in R^2 \mid z_2 \geq \frac{1}{z_1 + x}, z_1 \geq 0 \right\}.$$

Consider an open set

$$U := \left\{ (z_1, z_2) \in R^2 \mid z_2 > \frac{1}{2z_1}, z_1 \geq 0 \right\},$$

which includes the set $\mathbf{cl}\ F_6(0)$ but does not include any sets $F_6(x)$ for $x > 0$. This shows that F_6 is not weakly u.s.c. at $x = 0$ although it is similar to an u.s.c. map in image values.

To overcome this incompleteness, we introduce a more weaker notion of upper semicontinuity of set-valued maps. It is presented also in [1, p.39] when $F(x_0)$ is a compact set in Y.

Definition 2.6. (equally w-u.s.c.) Let Y be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally weak upper semicontinuous (equally w-u.s.c. for short) at x_0 if for any $\varepsilon > 0$ there exists a neighborhood V of x_0 such that $F(x) \subset B_Y(F(x_0), \varepsilon)$ for all $x \in V$, where $B_Y(F(x_0), \varepsilon) := \{y \in Y \mid d_Y(y, F(x_0)) < \varepsilon\}$.

Theorem 2.1. Let X and Y be a topological space and a metric space, respectively. If a set-valued map F from X into 2^Y is w-u.s.c., then it is also equally w-u.s.c. Conversely, if F is equally w-u.s.c. at x_0 and $\mathbf{cl}\ F(x_0)$ is a compact set, then it is w-u.s.c. at the point.

Proof. The first part is obvious. We prove only the second part. Let F be equally w-u.s.c. at x_0 and **cl** $F(x_0)$ a compact set, and let U be an open set including **cl** $F(x_0)$. For each $y \in \mathbf{cl}\ F(x_0)$, there is $\varepsilon(y) > 0$ such that $B_Y(y, \varepsilon(y)) \subset U$, and hence

$$\mathbf{cl}\ F(x_0) \subset \bigcup_{y \in \mathbf{cl}\ F(x_0)} B_Y(y, \varepsilon(y)/2) \subset U.$$

Since **cl** $F(x_0)$ is compact, there exist $y_1, \ldots, y_m \in \mathbf{cl}\ F(x_0)$ such that

$$\mathbf{cl}\ F(x_0) \subset \bigcup_{i=1}^{m} B_Y(y_i, \varepsilon(y_i)/2).$$

Let $\varepsilon^* := \min_{i=1,\ldots,m} \varepsilon(y_i) > 0$, then there is a neighborhood V of x_0 such that $F(x) \subset B_Y(F(x_0), \varepsilon^*/2)$ for all $x \in V$. Therefore, we have that $F(x) \subset U$ for all $x \in V$. In fact, let $z \in F(x)$, then it follows from $z \in B_Y(F(x_0), \varepsilon^*/2)$ that there exists $z^* \in F(x_0)$ such that $d_Y(z, z^*) < \varepsilon^*/2$. Since $z^* \in B_Y(y_{i_0}, \varepsilon(y_{i_0})/2)$ for some i_0, we have $d_Y(z, y_{i_0}) \leq d_Y(z, z^*) + d_Y(z^*, y_{i_0}) < \varepsilon(y_{i_0})$, and hence $z \in B_Y(y_{i_0}, \varepsilon(y_{i_0})) \subset U$. This completes the proof. ∎

As known from this theorem, whenever Y is a metric space and $F(x_0)$ is a compact set, three notions of u.s.c., w-u.s.c., and equally w-u.s.c. at x_0 are coincident with each other.

When Y is a topological vector space or more generally a topological group, the notion of equally w-u.s.c. is coincident with the following one.

Definition 2.7. (properly u.s.c.) Let Y be a topological vector space. A set-valued map $F : X \to 2^Y$ is said to be properly upper semicontinuous (p-u.s.c. for short) at x_0 if for any open neighborhood G of the origin θ, there exists a neighborhood V of x_0 such that $F(x) \subset F(x_0) + G$ for all $x \in V$.

Next, we provide another definition of equally w-u.s.c. in terms of nets (sequences).

Definition 2.8. (equally w-u.s.c.) Let Y be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally w-u.s.c. at x_0 if for any nets $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{y_\lambda\}$ with $y_\lambda \in F(x_\lambda)$, there exists a net (sequence) $\{z_\lambda\}$ such that $z_\lambda \in F(x_0)$ and $d_Y(z_\lambda, y_\lambda) \to 0$.

Theorem 2.2. Definitions 2.6. and 2.8. are coincident.

Proof. Assume that F is equally w-u.s.c. at x_0 defined by Definition 2.6.. Let $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{y_\lambda\}$ with $y_\lambda \in F(x_\lambda)$ be nets in X and Y, respectively. By the assumption, for $\varepsilon = 1/n$, $n = 1, 2, \ldots$, there exists a neighborhood V_n of x_0 such that $F(x) \subset B_Y(F(x_0), 1/n)$ for all $x \in V_n$. Since $x_\lambda \to x_0$, for each $n = 1, 2, \ldots$ there exists λ_n such that $x_\lambda \in V_n$ for all $\lambda \geq \lambda_n$, and hence $y_\lambda \in F(x_\lambda) \subset B_Y(F(x_0), 1/n)$ for all $\lambda \geq \lambda_n$. This means that $d_Y(y_\lambda, F(x_0)) < 1/n$ for all $\lambda \geq \lambda_n$. Therefore, we can take a net $\{z_\lambda\} \subset F(x_0)$ such that $d_Y(y_\lambda, z_\lambda) \to 0$.

Conversely, assume that F is equally w-u.s.c. at x_0 defined by Definition 2.8.. Suppose to the contrary that there are $\varepsilon_0 > 0$ and a net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and $F(x_\lambda) \not\subset B_Y(F(x_0), \varepsilon_0)$, which implies that there exists a net $\{y_\lambda\}$ such that $y_\lambda \in F(x_\lambda)$ and $d_Y(y_\lambda, F(x_0)) \geq \varepsilon_0$. By the assumption, we can take another net $\{z_\lambda\} \subset F(x_0)$ such that $d_Y(y_\lambda, z_\lambda) \to 0$, which is a contradiction to $d_Y(y_\lambda, F(x_0)) \geq \varepsilon_0$. This completes the proof. ∎

Now, we turn to the discrepancy between upper semicontinuity and closedness of set-valued maps. In [4, Th.3] and [3], uniformly compactness near x_0 guarantees the coincidence between upper semicontinuity and closedness at the point as follows: if F is uniformly compact near x_0, i.e., there is a neighborhood V of x_0 such that the closure of the set $\cup_{x \in V} F(x)$ is compact, then F is closed at x_0 if and only if $F(x_0)$ is compact and F is u.s.c. at the point. To observe this, we extend Hogan's closedness in Definition 2.3..

Definition 2.9. Let X and Y be two metric spaces. A set-valued map $F : X \to 2^Y$ is said to be weakly closed (w-closed for short) at x_0 if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $\{y_n\}$ with $y_n \in F(x_n)$, $y_n \to y_0$ for some $y_0 \in Y$ implies that $y_0 \in \mathbf{cl}\, F(x_0)$.

Remark 2.1. This definition is a slight extension of closedness, and a closed set-valued map is also w-closed. Conversely, if F is w-closed at x_0 and $F(x_0)$ is a closed set, then it is closed at the point. Moreover, we can verify, by using Definition 2.8. in terms of sequences, that any equally w-u.s.c. set-valued map is w-closed, and hence any u.s.c. map is also w-closed. If $F(x_0)$ is a closed set, then upper semicontinuity implies closedness.

Conversely, we can verify that closedness at x_0 implies upper semicontinuity at the point if the set-valued map is uniformly compact near x_0; see [4, Th.3]. Similarly, any w-closed map is also w-u.s.c. under the uniformly compactness, and therefore the three notions of weakly upper semicontinuity, equally weak upper semicontinuity, and weakly closedness at x_0 are equivalent with each other whenever the set-valued map is uniformly compact near the point.

Remark 2.2. We can verify that if a set-valued map F is closed and w-u.s.c. at x_0, then F is u.s.c. at the point. Actually, the graph of F is closed by [4, Th.2], and hence F is a closed-valued map, i.e., $F(x_0)$ is a closed set. Also, the maps F_3 in Example 2.2. and F_5 in Example 2.3. are w-closed and equally w-u.s.c. at $x = 0$ but not closed at the point. On the other hand, the map F_4 in Example 2.2. is closed at x_0 but not equally w-u.s.c. at the point, and the map F_6 in Example 2.4. is closed and equally w-u.s.c. but not w-u.s.c. at the point. Moreover, Example 2.6. shows the existence of a set-valued map which is equally w-u.s.c. but neither closed nor w-u.s.c.

Thus, the notion of weakly closedness of set-valued maps is a considerable large class of maps similar to upper semicontinuous maps.

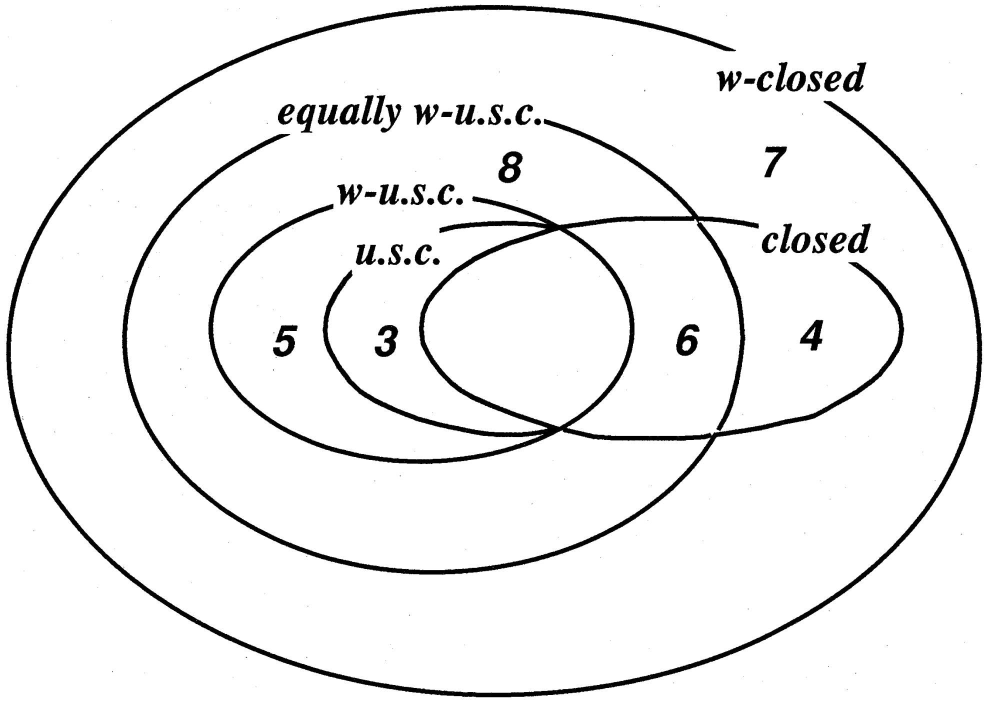

Figure 1: **Relationship among Various Upper Semicontinuities of Set-Valued Maps.**

Example 2.5. Let $X = Y = R$ and F_7 a set-valued map from R into 2^R defined by

$$F_7(x) := \begin{cases} \{y \in R \mid 0 \le y \le 1\} & \text{for } x < 0; \\ \{y \in R \mid 0 \le y < 1\} & \text{for } x = 0; \\ \{y \in R \mid \frac{1}{x} \le y \le \frac{1}{x} + 1\} & \text{for } x > 0. \end{cases}$$

One can verify that F_7 is w-closed at $x = 0$ but neither closed nor equally w-u.s.c. at the point.

Example 2.6. Let $X = R_+$, $Y = R^2$, and F_8 a set-valued map from R_+ into 2^{R^2} defined by

$$F_8(x) := \left\{(z_1, z_2) \in R^2 \mid z_2 > \frac{1}{z_1 + x}, z_1 \ge 0\right\}.$$

One can verify that F_8 is equally w-u.s.c. at $x = 0$ but neither closed nor w-u.s.c. at the point.

We conclude this chapter with an illustration of the inclusion structure shown in Figure 1 where the number stands for that of each set-valued map in Examples 2.2. to 2.6..

3. Theoretically Duality between Upper Semicontinuity and Lower Semicontinuity

We shall consider the possibility of extension of Berge's lower semicontinuity and Hogan's lower semicontinuity corresponding to the extended upper semicontinuities in Section 2.

First, we begin with Berge's lower semicontinuity. Let X and Y be two topological spaces, and $F : X \to 2^Y$. Since conditions $F(x_0) \cap U \neq \emptyset$ and $\mathbf{cl}\ F(x_0) \cap U \neq \emptyset$ are coincident for each open set $U \subset Y$, we can not extend lower semicontinuity into weaker notions in a similar way to that of Section 2. Then, we provide an improve notion of lower semicontinuity corresponding to equally weak upper semicontinuity in Definition 2.6.. The improved lower semicontinuity is precisely the dual concept of equally weak upper semicontinuity, and also stronger than lower semicontinuity.

Definition 3.1. (equally w-l.s.c.) Let Y be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally weak lower semicontinuous (equally w-l.s.c. for short) at x_0 if for any $\varepsilon > 0$ there exists a neighborhood V of x_0 such that $F(x_0) \subset B_Y(F(x), \varepsilon)$ for all $x \in V$.

Theorem 3.1. Let X and Y be a topological space and a metric space, respectively. If a set-valued map F from X into 2^Y is equally w-l.s.c., then it is also l.s.c. Conversely, if F is l.s.c. at x_0 and $\mathbf{cl}\ F(x_0)$ is a compact set, then it is equally w-l.s.c. at the point.

Proof. To prove the first part, let U be an open set satisfying $F(x_0) \cap U \neq \emptyset$. Suppose to the contrary that there is a net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and $F(x_\lambda) \cap U = \emptyset$. Hence there exist a vector $y_0 \in F(x_0)$ and a scalar $\varepsilon_0 > 0$ such that $B_Y(y_0, \varepsilon_0) \subset U$, and so $y_0 \notin B_Y(F(x_\lambda), \varepsilon_0)$. By the assumption, there exists a neighborhood V of x_0 such that $F(x_0) \subset B_Y(F(x), \varepsilon_0)$ for all $x \in V$. Since $x_\lambda \to x_0$, there exists λ_0 such that $x_\lambda \in V$ for all $\lambda \geq \lambda_0$, and hence $F(x_0) \subset B_Y(F(x_\lambda), \varepsilon_0)$ for all $\lambda \geq \lambda_0$, which is a contradiction to $y_0 \notin B_Y(F(x_\lambda), \varepsilon_0)$.

Next, we prove the second part. Let $\varepsilon > 0$, and then

$$\mathbf{cl}\ F(x_0) \subset \bigcup_{y \in \mathbf{cl}\ F(x_0)} B_Y(y, \varepsilon/2).$$

Since $\mathbf{cl}\ F(x_0)$ is a compact set, there exist $y_1, \ldots, y_m \in \mathbf{cl}\ F(x_0)$ such that

$$\mathbf{cl}\ F(x_0) \subset \bigcup_{i=1}^{m} B_Y(y_i, \varepsilon/2).$$

Let $U_i := B_Y(y_i, \varepsilon/2)$, then $F(x_0) \cap U_i \neq \emptyset$ for all $i = 1, \ldots, m$. By the assumption, there are neighborhoods $V_1, \ldots, V_m$ of x_0 such that $F(x) \cap U_i \neq \emptyset$ for all $x \in V_i$, $i = 1, \ldots, m$. Let $V := \bigcap_{i=1}^m V_i$, then we have that $F(x_0) \subset B_Y(F(x), \varepsilon)$ for all $x \in V$. In fact, let $z \in F(x_0)$, then it follows from $z \in \bigcup_{i=1}^m B_Y(y_i, \varepsilon/2)$ that there exists i_0 such that $z \in U_{i_0}$, i.e., $d_Y(z, y_{i_0}) < \varepsilon/2$. By $x \in V$, we have $F(x) \cap U_{i_0} \neq \emptyset$, which implies that $d_Y(y_{i_0}, z^*) < \varepsilon/2$ for some $z^* \in F(x)$. Therefore, we have $d_Y(z, z^*) \leq d_Y(z, y_{i_0}) + d_Y(y_{i_0}, z^*) < \varepsilon$. This completes the proof. ∎

We can provide another definition of equally w-l.s.c. in terms of nets (sequences), which is verified to be equivalent to Definition 3.1. in the same way as the proof of Theorem 2.2..

Definition 3.2. (equally w-l.s.c.) Let Y be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally w-l.s.c. at x_0 if for any nets $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{z_\lambda\}$ with $z_\lambda \in F(x_0)$, there exists a net (sequence) $\{y_\lambda\}$ such that $y_\lambda \in F(x_\lambda)$ and $d_Y(y_\lambda, z_\lambda) \to 0$.

This notion is precisely the dual concept of equally weak upper semicontinuity in terms of nets; see Definition 2.8.. Moreover, we can verify easily that the notion is stronger than openness (Hogan's lower semicontinuity); an equally w-u.s.c. set-valued map is also open, and conversely if F is open at x_0 and $\mathbf{cl}\, F(x_0)$ is a compact set, then it is equally w-u.s.c. at the point. When Y is a topological vector space or more generally a topological group, the notion of equally w-l.s.c. is coincident with the following one.

Definition 3.3. (properly l.s.c.) Let Y be a topological vector space. A set-valued map $F : X \to 2^Y$ is said to be properly lower semicontinuous (p-l.s.c. for short) at x_0 if for any open neighborhood G of the origin θ, there exists a neighborhood V of x_0 such that $F(x_0) \subset F(x) + G$ for all $x \in V$.

Finally, we obtain a theoretical conformable duality between upper semicontinuity and lower semicontinuity in terms of both neighborhoods and sequences in a metric space. Figure 2 illustrates the duality and the relationship among various semicontinuities of set-valued maps. From the figure we can observe that classical upper semicontinuity is precisely the dual concept of classical lower semicontinuity whenever the set-valued map is compact-valued in a metric space.

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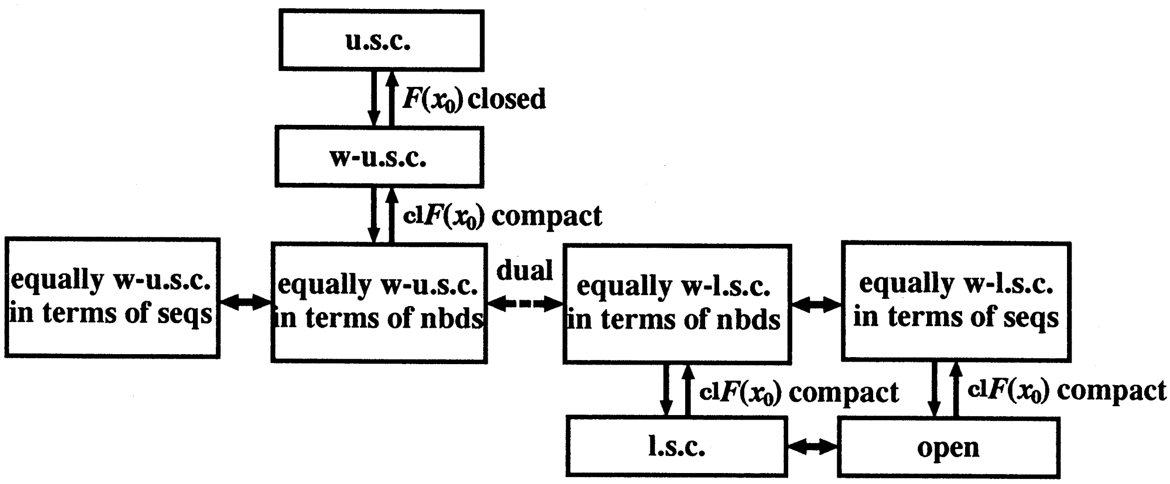

Figure 2: **Theoretical Duality between Upper Semicontinuity and Lower Semicontinuity.**

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